

Coping with NP-completeness: Special Cases

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Advanced Algorithms and Complexity
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The fact that a problem is **NP**-complete does not exclude an efficient algorithm for special cases of the problem.

Outline

① 2-Satisfiability

② Independent Sets in Trees

This part

- Striking connection between strongly connected components of a graph and formulas in 2-CNF
- A linear time algorithm for 2-SAT

2-Satisfiability (2-SAT)

Input: A set of clauses, each containing at most two literals (that is, a 2-CNF formula).

Output: Find a satisfying assignment (if exists).

Example

- $(x \vee y)(\bar{z})(z \vee \bar{x})$ is satisfied by
 $x = 0, y = 1, z = 0$

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- Essentially, it says that ℓ_1 and ℓ_2 cannot be both equal to 0
- In other words, if $\ell_1 = 0$, then $\ell_2 = 1$ and if $\ell_2 = 0$, then $\ell_1 = 1$

Definition

Implication is a binary logical operation denoted by \Rightarrow and defined by the following truth table:

x	y	$x \Rightarrow y$
0	0	1
0	1	1
1	0	0
1	1	1

Definition

For a 2-CNF formula, its **implication graph** is constructed as follows:

- for each variable x , introduce two vertices labeled by x and \bar{x} ;
- for each 2-clause $(\ell_1 \vee \ell_2)$, introduce two directed edges $\bar{\ell}_1 \rightarrow \ell_2$ and $\bar{\ell}_2 \rightarrow \ell_1$
- for each 1-clause (ℓ) , introduce an edge $\bar{\ell} \rightarrow \ell$

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Encodes all implications imposed by the formula.

$$(\bar{x} \vee y)(\bar{y} \vee z)(x \vee \bar{z})(z \vee y)$$

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x

\bar{x}

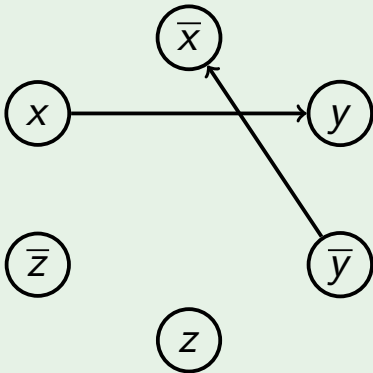
y

\bar{z}

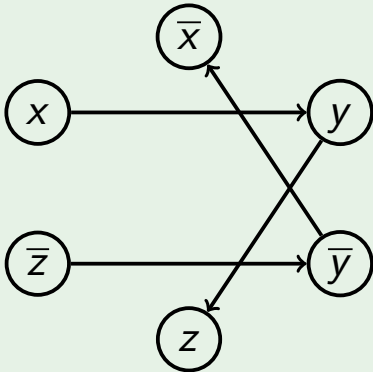
\bar{y}

z

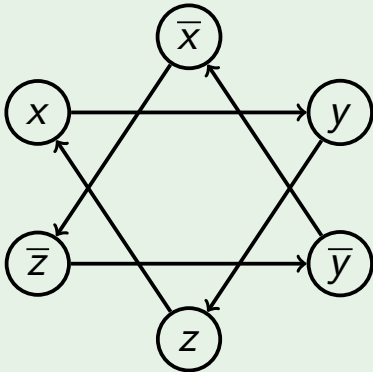
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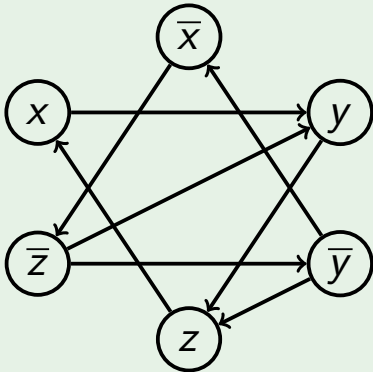
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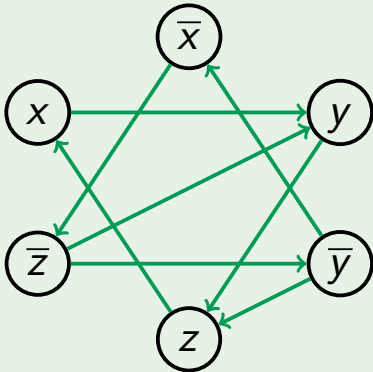
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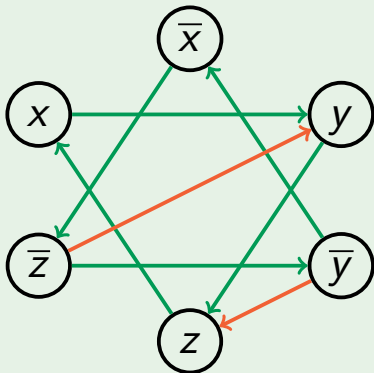


$$(\bar{x} \vee y)(\bar{y} \vee z)(x \vee \bar{z})(z \vee y)$$



$$x = 1, y = 1, z = 1$$

$$(\bar{x} \vee y)(\bar{y} \vee z)(x \vee \bar{z})(z \vee y)$$



$$x = 0, y = 0, z = 0$$

Thus, our goal is to assign truth values to the variables so that each edge in the implication graph is “satisfied”, that is, there is no edge from 1 to 0.

Skew-Symmetry

- The graph is skew-symmetric: if there is an edge $\ell_1 \rightarrow \ell_2$, then there is an edge $\bar{\ell}_2 \rightarrow \bar{\ell}_1$

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- This generalizes to paths: if there is a path from ℓ_1 to ℓ_2 , then there is a path from $\bar{\ell}_2$ to $\bar{\ell}_1$

Transitivity

Lemma

If all the edges are satisfied by an assignment and there is a path from ℓ_1 to ℓ_2 , then it cannot be the case that $\ell_1 = 1$ and $\ell_2 = 0$.

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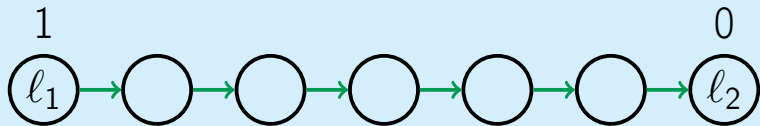


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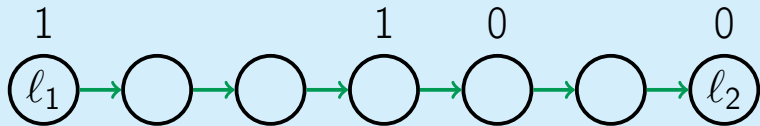


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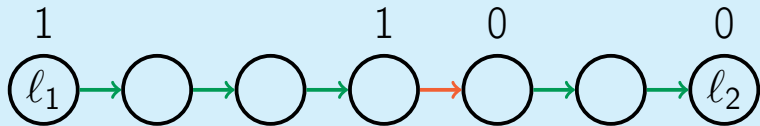


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- All variables lying in the same SCC of the implication graph should be assigned the same value

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- In particular, if a SCC contains a variable together with its negation, then the formula is unsatisfiable
- It turns out that otherwise the formula is satisfiable!

2SAT(2-CNF F)

construct the implication graph G

find SCC's of G

for all variables x :

 if x and \bar{x} lie in the same SCC of G :

 return “unsatisfiable”

find a topological ordering of SCC's

for all SCC's C in reverse order:

 if literals of C are not assigned yet:

 set all of them to 1

 set their negations to 0

return the satisfying assignment

2SAT(2-CNF F)

```
construct the implication graph  $G$ 
find SCC's of  $G$ 
for all variables  $x$ :
    if  $x$  and  $\bar{x}$  lie in the same SCC of  $G$ :
        return "unsatisfiable"
find a topological ordering of SCC's
for all SCC's  $C$  in reverse order:
    if literals of  $C$  are not assigned yet:
        set all of them to 1
        set their negations to 0
return the satisfying assignment
```

Running time: $O(|F|)$

Lemma

The algorithm 2SAT is correct.

Proof

- When a literal is set to 1, all the literals that are reachable from it have already been set to 1 (since we process SCC's in reverse topological order).

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Proof

- When a literal is set to 1, all the literals that are reachable from it have already been set to 1 (since we process SCC's in reverse topological order).
- When a literal is set to 0, all the literals it is reachable from have already been set to 0 (by skew-symmetry). □

Outline

① 2-Satisfiability

② Independent Sets in Trees

Planning a company party

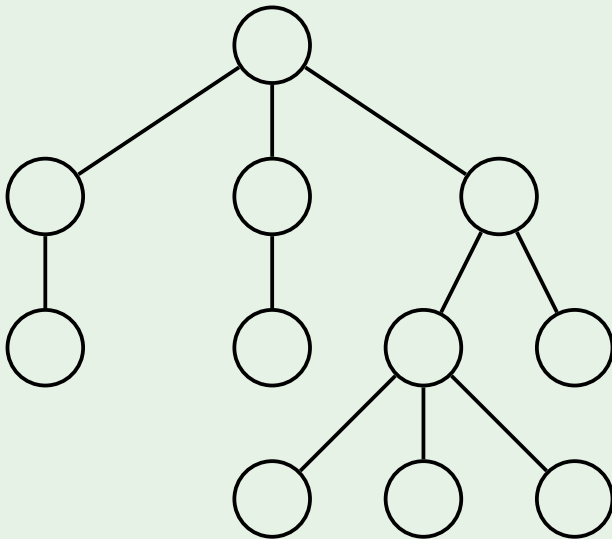
You are organizing a company party. You would like to invite as many people as possible with a single constraint: no person should attend a party with his or her direct boss.

Maximum independent set in a tree

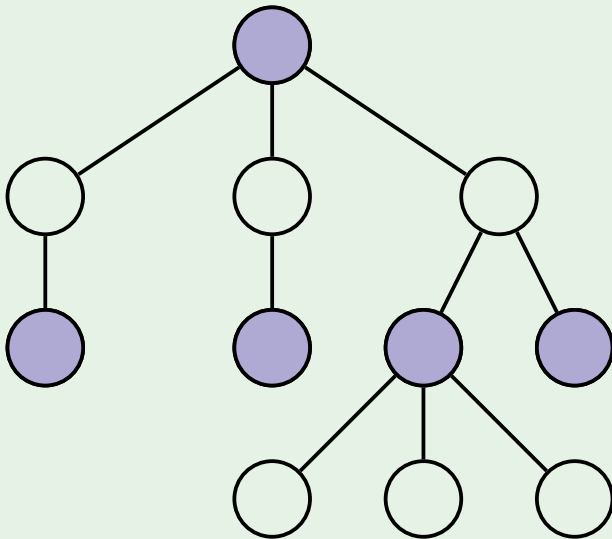
Input: A tree.

Output: An independent set (i.e., a subset of vertices no two of which are adjacent) of maximum size.

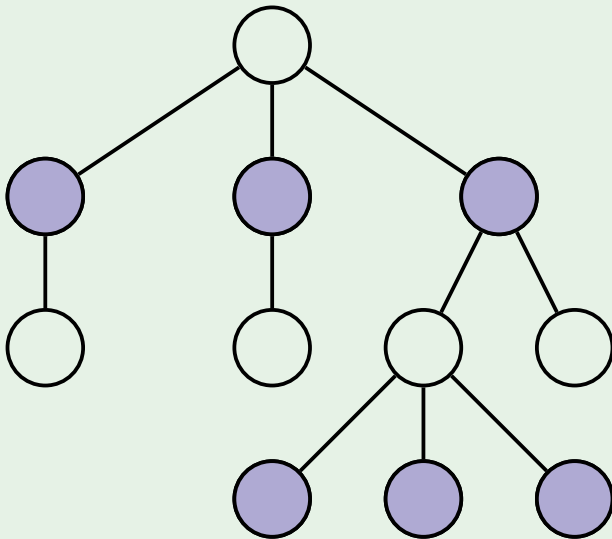
Example



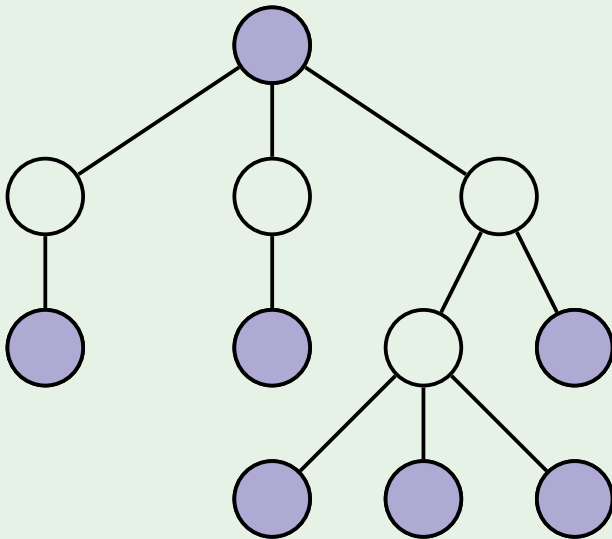
Example



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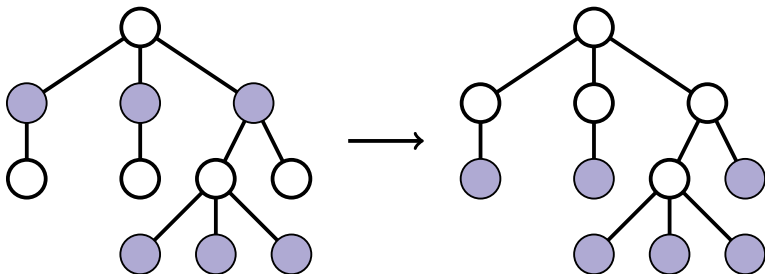


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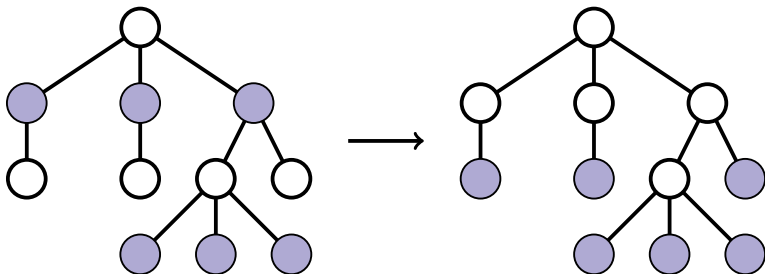
Safe move

For any leaf, there exists an optimal solution including this leaf.



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It is safe to take all the leaves.

PartyGreedy(T)

```
while  $T$  is not empty:  
    take all the leaves to the solution  
    remove them and their parents from  $T$   
return the constructed solution
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Running time: $O(|T|)$ (for each vertex, maintain the number of its children).

Planning a company party

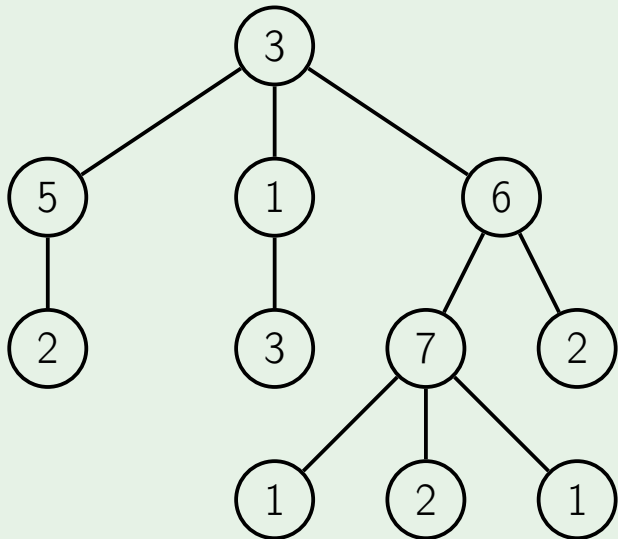
You are organizing a company party again. However this time, instead of maximizing the number of attendees, you would like to maximize the total fun factor.

Maximum weighted independent set in trees

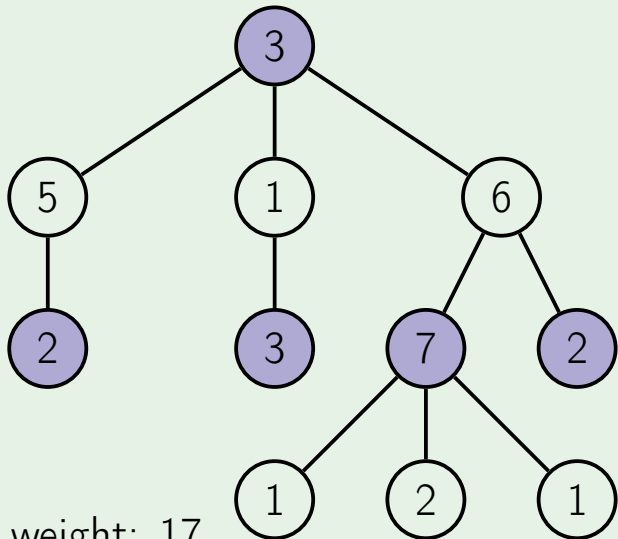
Input: A tree T with weights on vertices.

Output: An independent set (i.e., a subset of vertices no two of which are adjacent) of maximum total weight.

Example

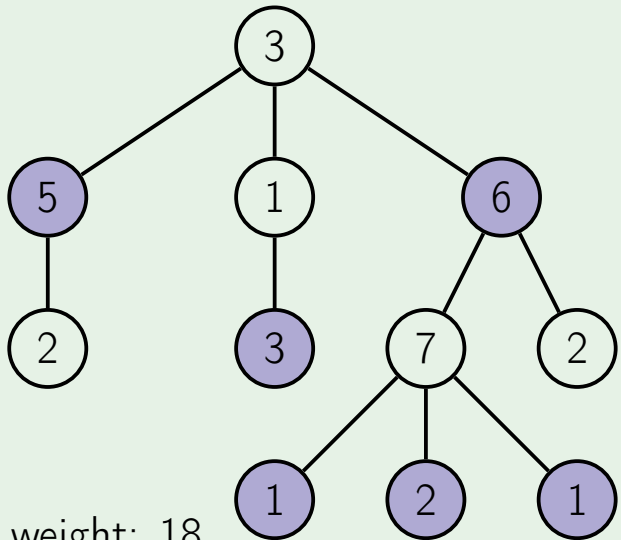


Example



total weight: 17

Example



total weight: 18

Subproblems

- $D(v)$ is the maximum weight of an independent set in a subtree rooted at v

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- $D(v)$ is the maximum weight of an independent set in a subtree rooted at v
- Recurrence relation: $D(v)$ is

$$\max \left\{ w(v) + \sum_{\substack{\text{grandchildren} \\ w \text{ of } v}} D(w), \sum_{\substack{\text{children} \\ w \text{ of } v}} D(w) \right\}$$

Function FunParty(v)

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if  $D(v) = \infty$ :  
    if  $v$  has no children:  
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         $m_1 \leftarrow w(v)$   
        for all children  $u$  of  $v$ :  
            for all children  $w$  of  $u$ :  
                 $m_1 \leftarrow m_1 + \text{FunParty}(w)$ 
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         $m_0 \leftarrow 0$   
        for all children  $u$  of  $v$ :  
             $m_0 \leftarrow m_0 + \text{FunParty}(u)$ 
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        for all children  $u$  of  $v$ :  
             $m_0 \leftarrow m_0 + \text{FunParty}(u)$   
         $D(v) \leftarrow \max(m_1, m_0)$   
return  $D(v)$ 
```

Example

